INFORMATION TECHNOLOGIES IN GENERALIZING PLANE GEOMETRY PROBLEMS

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Abstract. *Interesting plane geometry problems are considered, namely Olympiad problems of high quality and content, proposing possibilities for generalization and further investigations. In a corresponding research process accompanying results appear in a natural way ([1, 6]). The problems that are described refer to geometric configurations with the participation of circles. The generalizations relate to their transformation into second degree curves (mainly conics). In some cases, the initial circles are included in more general classes by means of the Geometer's Sketchpad program (GSP) which is applied as a heuristic tool for the purpose. Two approaches are used: 1) suitable examination of the problem conditions ([4]); 2) analysis of the solutions under consideration ([5]). In several cases interesting properties of the initial geometric relations are found.*

Key words: Triangle, Circle, Conic.

Generalization of basic elements in problem conditions

1. A problem for the orthocenter. The following IMO problem is considered:

Problem 1. Given is an acute $\triangle ABC$ with orthocenter H. The circle through H, whose center is the midpoint of the side BC, meets the sideline BC at A_1 and A_2. Analogously, the circle through H, whose center is the midpoint of the side CA, meets the sideline CA at B_1 and B_2. Also, the circle through H, whose center is the midpoint of the side AB, meets the sideline AB at C_1 and C_2. Prove that the points A_1, A_2, B_1, B_2, C_1 and C_2 are co-cyclic (Figure 1).

Consider an arbitrary $\triangle ABC$, for which the points M_a, M_b and M_c are the midpoints of the sides BC, CA and AB, respectively.

2. Curves, that are generated by an arbitrary point in the plane of a triangle. According to Problem 1, the points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic. Note that the circle is a special second-order curve. Now replace the point H by an arbitrary point P in the plane of $\triangle ABC$. Using a similar construction as in Problem 1, we get 6 points A_1, A_2, B_1, B_2, C_1 and C_2, which induce the following theorem (still a hypothesis for the moment) (Figure 2):

Theorem 1. (analogous to Problem 1) Given is a triangle ABC and an arbitrary point P in its plane. The circle through P, whose center is the midpoint of the side BC, meets the sideline BC at A_1 and A_2. Analogously, the circle through P, whose center is the midpoint of the side CA, meets the sideline CA at B_1 and B_2. Also, the circle through P, whose center is the midpoint of the side AB, meets the sideline AB at C_1 and C_2. Then, the points A_1, A_2, B_1, B_2, C_1 and C_2 lie on a second-order curve $k(P)$, which turns out to be a circle if and only if P is the orthocenter of $\triangle ABC$ (Figure 2).

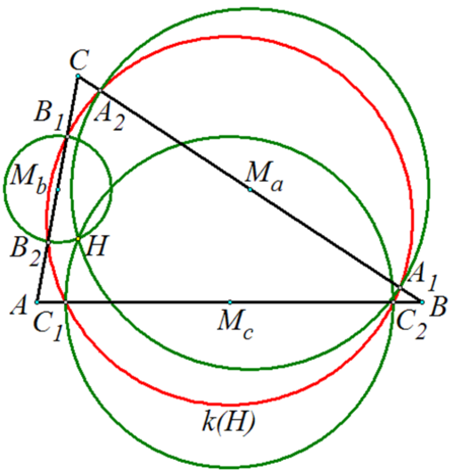

Figure 2

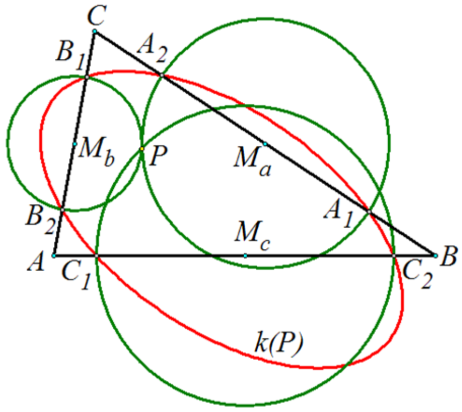

Figure 2

In what follows the notation $k(X)$ will be used for a second-order curve with center X (if the curve is central). Additionally, experiments with GSP suggest the following:

Corollary 1. If P is on the circumcircle Γ of $\triangle ABC$, then $k(P)$ is a hyperbola or it degenerates into two perpendicular lines (the set of the two lines is a special second-order curve).

3. Curves, that are generated by isogonal conjugate points with respect to a triangle. Note that the midpoints M_a, M_b and M_c, which are circle centers in Problem 1, are the orthogonal projections of the circumcenter O onto the sidelines of $\triangle ABC$, while the common point of the three circles is the orthocenter H. Let us turn down the situation, i. e. let the orthogonal projections of the orthocenter H be centers of the three circles passing through the circumcenter O. Now, the corresponding 6 points on the sidelines are co-cyclic. Something more, the GSP experiment shows that the new circle coincides with the previous one. Since H and O are isogonal conjugate with respect to $\triangle ABC$, then one could consider any other isogonal conjugate pair. We come to another assertion (again still a hypothesis for the moment):

Theorem 2. Let P^1 and P^2 be isogonal conjugate points with respect to a given $\triangle ABC$, while the points P_a^j, P_b^j and P_c^j be the orthogonal projections of P^j ($j = 1, 2$) onto the sidelines BC, CA and AB, respectively. The circle through P^s ($j \neq s = 1, 2$) with center P_a^j meets the sideline BC in A_1^j and A_2^j. The pairs B_1^j, B_2^j and C_1^j, C_2^j are defined on the sidelines CA and AB respectively in a similar way. Then, the points A_1^j, A_2^j, B_1^j, B_2^j, C_1^j and C_2^j ($j = 1, 2$) lie on equal circles with centers P^1 and P^2 (Figure 3).

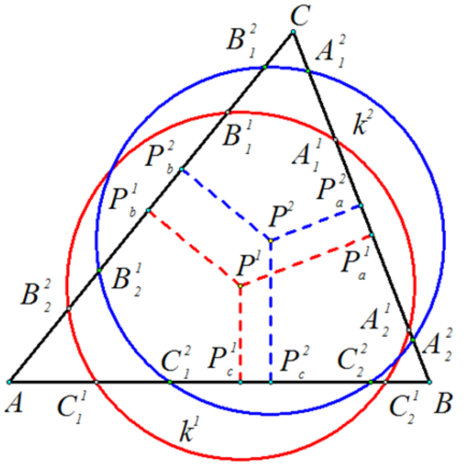

Figure 3

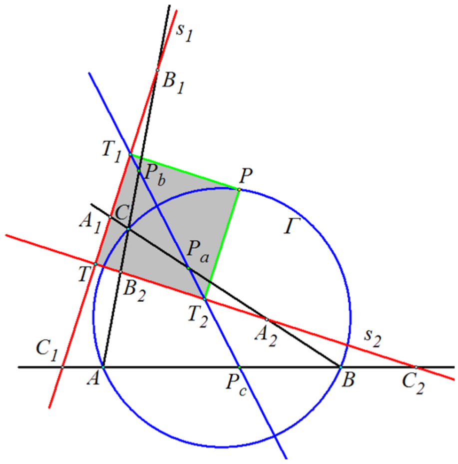

Figure 4

Of course, the points on the circumcircle Γ of $\triangle ABC$ (and only they) do not satisfy Theorem 2, since they have no isogonal conjugate ones with respect to $\triangle ABC$ in the sense of finite points. In this case the "defect" could be removed as in the Corollary 1. Construct the circles through $P \in \Gamma$ with centers P_a, P_b and P_c. Consider the intersection points of the circles with the sidelines BC, CA and AB in the way it is accomplished in Problem 1, Theorem 1 and Theorem 2. We state the following:

Theorem 3. Let P be on the circumcircle Γ of $\triangle ABC$, while P_a, P_b and P_c be its orthogonal projections onto the sidelines BC, CA and AB, respectively. The circle through P with center P_a meets the sideline BC in A_1 and A_2. Analogously, the circle through P, with center P_b meets the sideline CA in B_1 and B_2. Also, the circle through P with center P_c meets the sideline AB in C_1 and C_2. Then, the points A_1, A_2, B_1, B_2, C_1 and C_2 lie on two perpendicular lines s_1 and s_2 (Figure 4).

It is well-known that the points P_a, P_b and P_c from Theorem 3 are collinear and the line s_P on which they lie is known to be the Simson line. In connection with the Simson line note the following result:

Corollary 2. The intersection points of s_1, s_2 and s_P together with the generating point P define the vertices of a square (Figure 4).

PROOFS. We must legalize the formulated assertions in this paragraph by mathematical proofs. It is enough to limit ourselves to Theorem 1, since the applied technique is the same. Details could be found in the book [4]. Barycentric coordinates will be used with regard to $\triangle ABC$, namely $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$. Let $|BC| = a$, $|CA| = b$ and $|AB| = c$. Then $16S^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$, where S is the area of $\triangle ABC$. For an arbitrary point $P(\lambda,\, \mu, \nu)$ $(\lambda + \mu + \nu = 1)$ in the plane of $\triangle ABC$ consider the notation: $\delta = a^2\mu\nu + b^2\nu\lambda + c^2\lambda\mu$. The point P lies on the circumcircle of $\triangle ABC$ if and only if $\delta = 0$. Remember also that if $M_1(x_1,\, y_1,\, z_1)$ and $M_2(x_2,\, y_2,\, z_2)$ are points in the plane of $\triangle ABC$, then:

$$|M_1M_2|^2 = -(y_1-y_2)(z_1-z_2)a^2-(z_1-z_2)(x_1-x_2)b^2-(x_1-x_2)(y_1-y_2)c^2. \quad (1.1)$$

If the vectors $\vec{u_1}(\lambda_1,\, \mu_1,\, \nu_1)$ and $\vec{u_2}(\lambda_2,\, \mu_2,\, \nu_2)$ $(\lambda_1 + \mu_1 + \nu_1 = 0,\ \lambda_2 + \mu_2 + \nu_2 = 0)$ are coplanar to the plane of $\triangle ABC$, then they are perpendicular $(\vec{u_1} \perp \vec{u_2})$ if and only if the following equality is verified:

$$(\mu_1\nu_2 + \mu_2\nu_1)a^2 + (\nu_1\lambda_2 + \nu_2\lambda_1)b^2 + (\lambda_1\mu_2 + \lambda_2\mu_1)c^2 = 0. \quad (1.2)$$

Proof of Theorem 1. ([3, 4]) It is clear that the theorem is meaningless when P coincides with one of the vertices A, B and C. Such cases are excluded.

Let $p_a = \dfrac{|PM_a|}{a}$, $p_b = \dfrac{|PM_b|}{b}$ and $p_c = \dfrac{|PM_c|}{c}$. The coordinates of the 6 points under consideration are deduced from (1):

$$A_1\left(0,\, \frac{1}{2} + p_a,\, \frac{1}{2} - p_a\right), \quad A_2\left(0,\, \frac{1}{2} - p_a,\, \frac{1}{2} + p_a\right),$$
$$B_1\left(\frac{1}{2} - p_b,\, 0,\, \frac{1}{2} + p_b\right), \quad B_2\left(\frac{1}{2} + p_b,\, 0,\, \frac{1}{2} - p_b\right),$$
$$C_1\left(\frac{1}{2} + p_c,\, \frac{1}{2} - p_c,\, 0\right), \quad C_2\left(\frac{1}{2} - p_c,\, \frac{1}{2} + p_c,\, 0\right).$$

Substitute the coordinates and check the following equation:

$$k(P):\ \begin{array}{l} (4p_a^2 - 1)(4p_b^2 - 1)(4p_c^2 - 1)(x^2 + y^2 + z^2) \\ +2(4p_b^2 - 1)(4p_c^2 - 1)(4p_a^2 + 1)yz \\ 2(4p_c^2 - 1)(4p_a^2 - 1)(4p_b^2 + 1)zx \\ +2(4p_a^2 - 1)(4p_b^2 - 1)(4p_c^2 + 1)xy = 0. \end{array}$$

This proves that the points A_1, A_2, B_1, B_2, C_1 and C_2 lie on a second-order curve $k(P)$.

Remark. In the proof it is not used the fact that the circles under consideration are concurrent at a point. This suggests a possibility to examine cases: for example, the cases when they are tangent to a circle, tangent to a line or cases with other additional conditions.

It follows from that:

$$\begin{aligned}
|M_aP|^2 &= a^2p_a^2 = \frac{1}{2}(-a^2+b^2+c^2)\lambda + \frac{1}{4}a^2 - \delta,\\
|M_bP|^2 &= b^2p_b^2 = \frac{1}{2}(a^2-b^2+c^2)\mu + \frac{1}{4}b^2 - \delta,\\
|M_cP|^2 &= c^2p_c^2 = \frac{1}{2}(a^2+b^2-c^2)\nu + \frac{1}{4}c^2 - \delta.
\end{aligned}$$

In order to prove the second part of Theorem 1, denote by O_A and O_C the centers of the circumcircles of the triangles $A_1A_2B_1$ and $C_1C_2B_1$, respectively. Further, find (using (2)) the equations of two perpendicular bisectors of each triangle and consider the corresponding system. Thus, the coordinates of O_A and O_C could be determined as follows:

$$\begin{aligned}
x_{O_A} &= \frac{2a^2\left[4p_{ab}+2(-a^2+b^2+c^2)p_b-c^2\right]}{32S^2(1-p_b)},\\
y_{O_A} &= \frac{4(a^2+b^2-c^2)p_{ab}-4b^2(a^2-b^2+c^2)p_b+2a^2b^2+b^2c^2+c^2a^2-a^4-b^4}{32S^2(1-p_b)},\\
z_{O_A} &= \frac{4(a^2-b^2+c^2)p_{ab}-4c^2(a^2+b^2-c^2)p_b+2a^2b^2+3b^2c^2+c^2a^2-a^4-b^4-2c^4}{32S^2(1-p_b)},\\
x_{O_C} &= \frac{4(-a^2+b^2+c^2)p_{cb}-4a^2(-a^2+b^2+c^2)p_b+2b^2c^2+3a^2b^2+c^2a^2-b^4-c^4-2a^4}{32S^2(1-p_b)},\\
y_{O_C} &= \frac{4(-a^2+b^2+c^2)p_{cb}-4b^2(a^2-b^2+c^2)p_b+a^2b^2+2b^2c^2+c^2a^2-b^4-c^4}{32S^2(1-p_b)},\\
z_{O_C} &= -\frac{2c^2\left[4p_{cb}+2(a^2+b^2-c^2)p_b-a^2\right]}{32S^2(1-p_b)},
\end{aligned}$$

where $p_{ab} = a^2p_a^2 - b^2p_b^2$ and $p_{cb} = c^2p_c^2 - b^2p_b^2$.

If the circles coincide, then $O_A \equiv O_C$ and we deduce that:

$$\begin{aligned}
&-2a^2(-a^2+b^2+c^2)\lambda + (3a^2-b^2+c^2)(a^2-b^2+c^2)\mu\\
&\quad -(a^2-b^2+c^2)(a^2+b^2-c^2)\nu = 0,\\
&(-a^2+b^2+c^2)(a^2+b^2-c^2)\lambda + 2(c^2-a^2)(a^2-b^2+c^2)\mu\\
&\quad -(a^2+b^2-c^2)(-a^2+b^2+c^2)\nu = 0.
\end{aligned}$$

Add the condition $\lambda + \mu + \nu = 1$, thus obtaining a linear system of three equations with three unknowns:

$$\lambda_H \frac{(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)}{16S^2},$$
$$\mu_H \frac{(a^2 + b^2 - c^2)(-a^2 + b^2 + c^2)}{16S^2},$$
$$\nu_H \frac{(-a^2 + b^2 + c^2)(a^2 - b^2 + c^2)}{16S^2}.$$

The result coincides with the coordinate representation of the orthocenter H. Finally, note that the unique solution of the system, describing the relation $O_A \equiv O_C$, implies that H is unique, so that $k(P \equiv H)$ is a circle. This ends the proof.

Further details are included in the book [4].

References

[1] S. Grozdev, *For High Achievements in Mathematics. The Bulgarian Experience (Theory and Practice)*, Sofia: Association for the development of education, 2007, ISBN: 978-954-92139-1-1

[2] S. Grozdev, V. Nenkov, A property of central conics associated with a triangle, *Mathematics and Mathematics Education*, 2011, 40, pp. 394–399, ISSN: 1313-3330

[3] S. Grozdev, V. Nenkov, *Around the orthocenter in the plane and the space*, Sofia: Archimedes, 2012, ISBN: 978-954-779-145-9

[4] S. Grozdev, V. Nenkov, Loci generated by conjugate lines and cevians, *Mathematics and Informatics*, 2012, 6, pp. 562–577, ISSN: 1310-2230

[5] S. Grozdev, V. Nenkov, Three properties of the triangle, generated by its altitudes, *Mathematics and Informatics*, 2013, 2, pp. 176–183, ISSN: 1310-2230

[6] V. Nenkov, *Increasing mathematical competence by dynamic geometry*, Sofia: Archimedes, 2020, ISBN: 978-954-779-291-3

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