TRIPLED FIXED POINTS IN MODULAR FUNCTION SPACES FOR ρ-CONTRACTIVE MAPS

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Abstract. *This paper investigates tripled fixed points for contractive type mappings in modular function spaces. Building on the classical framework of modular function spaces introduced by Kozłowski, we extend the theory of multi-tupled fixed points to the setting of regular convex function modulars. We establish existence and uniqueness results for tripled fixed points of ρ-contractive mappings and provide a constructive iterative scheme converging to the unique solution. The proofs rely on the modular completeness of L_ρ, the Δ_2-property, and the uniform continuity of the modular. In addition, we highlight the connection between tripled fixed points and fixed points of induced maps, following the framework of Petrușel.*

Key words: Tripled fixed points, modular function spaces, contraction mappings

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1. Introduction

Modular function spaces provide a natural setting to extend classical fixed point theory beyond normed spaces initiated by Kozłlowski in [8] and by Musielak in [10], unifying and generalizing classical Banach space structures. The theory of fixed point in modular function space was initiated by Khamsi, Kozłlowski and Reich in [6]. Important results on fixed points in modular function spaces and the related geometric properties of these spaces can be found in [7].

Tripled fixed points and tripled best proximity point were introduced in [4]. Some recent applications of the notion of tripled fixed points were presented in [1] for solving of systems of nonlinear matrix equations and in [2] about an investigation of equilibrium in oligopoly markets with three dominating players.

2. Preliminary Remarks

For the sake of brevity, we recall only those fundamental notions and results from the theory of modular function spaces that are directly relevant to the subsequent results. For a more complete background on the theory of modular function spaces, we refer the reader to [7], which concepts and notations we will follow.

Let Ω be a nonempty set and Σ be a non-trivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup_n K_n$.

By $\mathcal{E}$ we denote the linear space of all simple functions whose supports belong to $\mathcal{P}$. By $\mathcal{M}_\infty$ we denote the space of all extended measurable functions, i.e. all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $g_n \subset \mathcal{E}$, with $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

For $A \subset \Omega$ we write $\mathbf{1}_A$ for the characteristic function of A.

Definition 2.1. *([7]) Let $\rho : \mathcal{M}_\infty \to [0, \infty)$ be a non-trivial convex and even function. We say that ρ is a regular convex function pseudomodular if*

1. *$\rho(0) = 0$;*
2. *if $|f(\omega)| \leq |g(\omega)|$ for all $\omega \in \Omega$, then $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;*
3. *there holds $\rho\big(f\mathbf{1}_{A\cup B}\big) \leq \rho\big(f\mathbf{1}_A\big) + \rho\big(f\mathbf{1}_B\big)$ for $A, B \in \Sigma$ with $A \cap B \neq \varnothing$ and $f \in \mathcal{M}_\infty$;*
4. *if $f_n(\omega) \uparrow |f(\omega)|$ for all $\omega \in \Omega$, then $\rho(f_n) \uparrow \rho(f)$, where $f_n \in \mathcal{M}_\infty$;*
5. *if $g_n \in \mathcal{E}$ and $|g_n(\omega)| \downarrow 0$, then $\rho(g_n) \downarrow 0$.*

If $\lim\limits_{n\to\infty} \rho(x_n - x) = 0$, and $\{x_n\}n = 1^\infty$ be a bounded sequence then for any $y \in L_\rho$ there holds $\rho(x - y) = \lim\limits_{n\to\infty} \rho(x_n - y)$.

Analogously to the measure-space setting, we say that a set $A \in \Sigma$ is ρ-null if $\rho(g\mathbf{1}_A) = 0$ for every $g \in \mathcal{E}$. We say that a property holds ρ-almost everywhere, if the exceptional set is ρ-null. As usual, we identify measurable sets whose symmetric difference is ρ-null as well as measurable functions that differ only on a ρ-null set. With this convention we set where each $f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ is regarded as an equivalence class of functions equal ρ-a.e. When no ambiguity arises we write simply $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

Definition 2.2. *([7]) We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0$, ρ-a.e.*

The class of all non-zero regular convex function modulars defined on Ω will be denoted by $\mathfrak{R}$.

Definition 2.3. *([7]) Let $\rho \in \Re$. We say that $\rho \in \mathcal{R}$ is uniformly continuous if for any $L > 0$ and any $\varepsilon > 0$ there exists $\delta(L, \varepsilon) > 0$, such that if $\rho(x) \le L$ and $\rho(y) < \delta$ there holds the inequality $|\rho(y + x) - \rho(x)| < \varepsilon$.*

Definition 2.4. *([7]) Let $\rho : \mathcal{M} \to [0, \infty)$ be a convex function modular. The associated modular function space is $L_\rho = \{ f \in \mathcal{M} : \lim_{\lambda \to 0} \rho(\lambda f) = 0\}$, equipped with the Luxemburg norm $\|f\|_\rho = \inf\left\{\alpha > 0 : \ \rho\left(\frac{f}{\alpha}\right) \le 1\right\}$.*

Theorem 2.1. *([7]) Let ρ be a convex modular. Then:*

1. *$(L_\rho, \|\cdot\|_\rho)$ is complete;*
2. *if $\lim_{n\to\infty} \rho(\alpha f_n) = 0$ for some $\alpha > 0$, then there is a subsequence of $\{f_n\}$ that converges ρ–a.e. to 0;*
3. *if $f_n \to f$ ρ–a.e., then $\rho(f) \le \liminf \rho(f_n)$.*

Definition 2.5. *([7]) Let L_ρ be a modular function space.*

1. *$f_n \to f$ (ρ) if $\rho(f_n - f) \to 0$;*
2. *$\{f_n\}$ is ρ–Cauchy if $\rho(f_m - f_n) \to 0$ as $m, n \to \infty$;*
3. *L_ρ is ρ–complete if every ρ–Cauchy sequence is ρ–convergent;*
4. *$A \subset L_\rho$ is ρ–closed if the limit f every ρ–convergent sequence $\{f_n\}_{n=1}^{\infty} \subset A$ belong to A;*
5. *$A \subset L_\rho$ is ρ–bounded if there is $f \in L_\rho$ and a constant M such that $\rho(f, g) < M$ for every $g \in A$;*
6. *ρ has the Δ_2–property if $\rho(f_n) \to 0$ implies $\rho(2f_n) \to 0$.*

Theorem 2.2. *([7]) The space (L_ρ, ρ) is ρ–complete.*

Theorem 2.3. *([7]) If ρ satisfies the Δ_2–property, then ρ–convergence is equivalent to norm convergence in L_ρ.*

3. Tripled Fixed Points for ρ Contractive Mappings in Modular Function Spaces

The notion of coupled fixed points have been introduced in [5]. Late a more concise version have been presented in [3] and [9], which can be considered as the starting point for the investigations in the theory of coupled, tripled and n-tupled fixed points. The introduction of n-tupled fixed point is done in [12]. Following [12], we will recall the n-tupled fixed point definition in the particular case of tripled fixed points.

Definition 3.1. *([12]) Let A be a non-empty subset of a modular function space X and $F\colon A^3 \to A$. An ordered triple $(x, y, z) \in A^3$ is a tripled fixed point of F in A if $x = F(x, y, z)$, $y = F(y, z, x)$, and $z = F(z, x, y)$.*

It is worth to mention that a deep observation in [11] presents a connection between coupled fixed points and fixed points. We will rewrite it in the context of tripled fixed points. Let $F\colon A^3 \to A$ and $T : A^3 \to A^3$ be defined as $T(x, y, z) = (F(x, y, z), F(y, z, x), F(z, x, y))$, then the ordered triple (x, y, z) is a tripled fixed point for (F) if and only if it is a fixed point for T. For the rest of the paper we will use the notation T for the functions defined just above.

The next notation will be useful for fitting some formulas into the text field and simplifying them.

Definition 3.2. *Let $A \subset X$ be non-empty and $F\colon A^3 \to A$. For any initially chosen ordered triple of points $(x_0, y_0, z_0) \in A^3$ we will consider the iteratively constructed sequences $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}, \{z_n\}_{n=0}^{\infty} \subset A$, defined by $x_{n+1} = F(x_n, y_n, z_n)$, $y_{n+1} = F(y_n, z_n, x_n)$, $z_{n+1} = F(z_n, x_n, y_n)$ or equivalently by $(x_{n+1}, y_{n+1}, z_{n+1}) = T(x_n, y_n, z_n)$ for $n \geq 0$.*

In what follows we will always assume that the considered iterated sequences $\{x_n\}, \{y_n\}, \{z_n\} \subset A$ are the one defined in Definition 3.2

Let us introduce a notation: for (x, y, z) we will denote $\pi_0(x, y, z) = (x, y, z)$, $\pi_1(x, y, z) = (y, z, x)$, and $\pi_2(x, y, z) = (z, x, y)$.

Definition 3.3. *Let $A \subset X$ be non-empty and $F\colon A^3 \to A$. We call F a ρ–contraction map if there exists $\alpha \in \left[0, \frac{1}{2}\right)$ such that, for all $x, y, z, u, v, w \in A$,*

$$\begin{aligned} P_1 = \sum_{k=0}^{2} \rho\big(F(\pi_k(x, y, z)) - F(\pi_k(u, v, w))\big) \\ \leq \alpha\big(\rho(x - u) + \rho(y - v) + \rho(z - w)\big). \end{aligned}$$

Theorem 3.1. *Let $\rho \in \mathfrak{R}$ be uniformly continuous and let $A \subset L_\rho$ be non-empty, ρ-closed, and ρ-bounded. If $F\colon A^3 \to A$ is a ρ–contraction map, then*

1. *F possesses a unique tripled fixed point $(x, y, z) \in A^3$;*
2. *for any initially chosen triple $(x_0, y_0, z_0) \in A^3$ the iterative sequences sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{z_n\}_{n=0}^{\infty}$ converge (in the modular sense) to that unique tripled fixed point (x, y, z), respectively.*

Proof. Let $(x_0, y_0, z_0) \in A^3$ be an arbitrary initial triple and consider the sequences given by the iteration scheme: $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{z_n\}_{n=0}^{\infty}$

Using the ρ-contraction property, for each $n \geq 1$, we have:

$$\begin{aligned} P_2 &= \rho(x_{n+1} - x_n) + \rho(y_{n+1} - y_n) + \rho(z_{n+1} - z_n) \\ &= \sum_{k=0}^{2} \rho(F(\pi_k(x_n, y_n, z_n)) - F(\pi_k(x_{n-1}, y_{n-1}, z_{n-1}))) \\ &\leq \alpha\big(\rho(x_n - x_{n-1})\big). \end{aligned}$$

We define $S_n(x_0, y_0, z_0) = \rho(x_{n+1} - x_n) + \rho(y_{n+1} - y_n) + \rho(z_{n+1} - z_n)$, a notation that we will use till the end of the paper.

Summing the last three inequalities, we obtain the inequality

$$S_n(x_0, y_0, z_0) \leq \alpha S_n(x_0, y_0, z_0),$$

and inductively $S_n(x_0, y_0, z_0) \leq \alpha^n S_0(x_0, y_0, z_0)$.

Therefore $\lim\limits_{n\to\infty} S_n(x_0, y_0, z_0) = 0$.

We put $W_{n,m}(x_0, y_0, z_0) = \rho(x_{n+p} - x_n) + \rho(y_{n+p} - y_n) + \rho(z_{n+p} - z_n)$, a notation that we will use till the end of the paper.

Next, we show the sequences are Cauchy. Indeed, for arbitrary $n, p \in \mathbb{N}$: $W_{n+p,n}(x_0, y_0, z_0) \leq \alpha^n W_{p,0}(x_0, y_0, z_0)$.

Since A is ρ-bounded, there exists $M > 0$ with $\rho(u - v) \leq M$ for all $u, v \in A$. Thus, $W_{n+p,n}(x_0, y_0, z_0) \leq 3M\alpha^n$. Given that $\alpha^n \to 0$, for any $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that for all $n \geq N$, $3M\alpha^n < \varepsilon$. Hence, the sequences are Cauchy.

By the ρ-completeness of L_ρ and ρ-closedness of A, there exist limits $(x, y, z) \in A^3$ satisfying:

$$\lim_{n\to\infty} \rho(x_n - x) = 0, \quad \lim_{n\to\infty} \rho(y_n - y) = 0, \quad \text{and} \quad \lim_{n\to\infty} \rho(z_n - z) = 0.$$

We now prove (x, y, z) is a tripled fixed point. Using the uniform continuity of ρ and the contraction condition we observe:

$$\begin{aligned} P_3 &= \rho(x - F(x, y, z)) + \rho(y - F(y, z, x)) + \rho(z - F(z, x, y)) \\ &= \lim_{n\to\infty} \big(\rho(x_{n+1} - F(x, y, z)) + \rho(y_{n+1} - F(y, z, x)) \\ &\quad + \rho(z_{n+1} - F(z, x, y))\big) \\ &= \lim_{n\to\infty} \sum_{k=0}^{2} \rho(F(\pi_k(x_n, y_n, z_n)) - F(\pi_k(x, y, z))) \\ &\leq \lim_{n\to\infty} \alpha(\rho(x_n - x) + \rho(y_n - y) + \rho(z_n - z)) = 0. \end{aligned}$$

From the assumption that $\alpha \in [0, 1/2)$ it follows that $\rho(x - F(x,y,z)) = \rho(y - F(y,z,x)) = \rho(z - F(z,x,y)) = 0$, i.e, $x = F(x,y,z)$, $y = F(y,z,x)$, and $z = F(z,x,y)$.

Hence (x,y,z) is a tripled fixed point for F.

To establish uniqueness, suppose (u,v,w) is another tripled fixed point. Then

$$\begin{aligned} P_3 &= \rho(x-u) + \rho(y-v) + \rho(w-z) \\ &= \sum_{k=0}^{2} \rho(F(\pi_k(x,y,z)) - F(\pi_k(u,v,w))) \\ &\leq \alpha(\rho(x-u) + \rho(y-v) + \rho(w-z)). \end{aligned}$$

Finally, convergence to the unique tripled fixed point (x,y,z) for arbitrary initial triple $(u_0,v_0,w_0) \in A^3$, $(u_0,v_0,w_0) \neq (x_0,y_0,z_0)$ it follows from the the proof as far as the iterated sequences will converge to a tripled fixed points, and by the uniquenes it should be (x,y,z).

Suppose that the tripled fixed point (x,y,z) do not consists with equal elements, i.e, $\rho(x-y) + \rho(y-z) + \rho(z-x) > 0$. From

$$\begin{aligned} P_4 &= \rho(x-y) + \rho(y-z) + \rho(z-x) \\ &= \sum_{k=0}^{2} \rho(F(\pi_k(x,y,z)) - F(\pi_k(y,z,x))) \\ &\leq \alpha(\rho(x-y) + \rho(y-z) + \rho(z-x)), \end{aligned}$$

it follows that $\rho(x-y) = \rho(y-z) = \rho(z-x) = 0$ and consequently the tripled fixed point of F is (x,x,x).

□

References

[1] A. Ali, C. Dinkova, H. Kulina, A. Ilchev, B. Zlatanov, Tripled fixed points, obtained by Ran-Reunrings theorem for monotone maps in partially ordered metric spaces, *Mathematics*, 2025, Vol. 13, No 5, 760, `doi:10.3390/math13050760`, ISSN: 22277390

[2] A. Ali, A. Ilchev, V. Ivanova, H. Kulina, B. Zlatanov, Modeling the tripodal mobile market using response functions instead of payoff maximization, *Mathematics*, 2025, Vol. 13, No. 1, 171, `doi:10.3390/math13010171`, ISSN: 22277390

[3] T. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Analysis, Theory, Methods and Applications*, 2006, Vol. 65, No. 7, 1379–1393, doi:10.1016/j.na.2005.10.017, ISSN: 0362546X

[4] M. Borcut, V. Berinde, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, *Applied Mathematics and Computation*, 2012, Vol. 16, No. 10, 5929–5936, doi:10.1016/j.amc.2011.11.049, ISSN: 00963003

[5] D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, *Nonlinear Analysis*, 1987, Vol. 11, No. 5, 623–632, doi:10.1016/0362-546X(87)90077-0, ISSN: 0362546X

[6] M. Khamsi, W. Kozłowski, S. Reich, Fixed point theory in modular function spaces, *Nonlinear Analysis*, 1990, Vol. 14, No. 11, 935–953, doi:10.1016/0362-546X(90)90111-S, ISSN: 0362546X

[7] M. Khamsi, W. Kozłowski, *Fixed point theory in modular function spaces*, Birkhäuser Cham, 2015, ISBN: 9783319140506, doi:10.1007/978-3-319-14051-3

[8] W. Kozłowski, Notes on modular function spaces. I, *Commentationes Mathematicae*, 1988, Vol. 28, No. 1, 87–100, http://eudml.org/doc/291891, ISSN: 2080-1211.

[9] V. Lakshmikantham, L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Analysis, Theory, Methods and Applications*, 2009, Vol. 70, No. 12, 4341–4349, doi:10.1016/j.na.2008.09.020, ISSN: 0362546X

[10] J. Musielak, *Orlicz spaces and modular spaces*; Springer Berlin, Heidelberg, Germany, 1983, ISBN: 9783540127062, doi:10.1007/BFb0072210

[11] A. Petruşel, Fixed points vs. coupled fixed points. *Journal of Fixed Point Theory and Applications*, 2018, **2018**, Vol. 20, No. 4, 150, doi:10.1007/s11784-018-0630-6, ISSN: 16617738

[12] B. Samet, C. Vetro, Coupled fixed point F-invariant set and fixed point of N-orders, *Annals of Functional Analysis*, 2010, Vol. 1, No. 2, 46–56, doi:10.15352/afa/1399900586, ISSN: 26397390

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