GENERALIZATION OF JOHN SMITH'S PROBLEM (THE PEPYS-NEWTON PROBLEM)

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Abstract. *We study a generalization of John Smith's problem, better known as the Pepys-Newton problem. The case of a generalized die with an arbitrary number of sides n is studied. Explicit expressions for the corresponding probabilities are derived, and numerical calculations are presented for small values of n. Using the Central Limit Theorem, a proof is given that the limiting probability equals $\frac{1}{2}$.*

Key words: Newton–Pepys Problem, Bernoulli Scheme, Central Limit Theorem, Poisson Approximation.

Introduction

The Newton–Pepys problem is a classical question in probability theory originating from a correspondence in 1693 between Samuel Pepys and Isaac Newton. The problem was posed to Pepys by a schoolteacher named John Smith and concerns the probability of obtaining a given number of sixes when throwing multiple dice.

Although the exact probabilities can be computed using elementary combinatorics, the problem is notable for the subtle comparison of competing probabilistic strategies and for Newton's attempt to provide an intuitive explanation of the correct answer.

Numerous historical and analytical discussions of the problem can be found in the literature (see, for example, [1–4]). Several authors have considered generalizations of the Newton–Pepys problem. In particular, it has been observed that in a natural asymptotic regime the relevant probabilities appear to converge to $\frac{1}{2}$, but a rigorous proof of this fact is often omitted or only sketched. The main contribution of the present paper is to provide such a proof using the Central Limit Theorem.

Definition of the problem

John Smith's problem, better known as Newton–Pepys problem, is a classical probability problem, concerns the probability of throwing sixes using a certain number of dice.

The problem is as follows:

A claims that he will obtain at least one six by throwing 6 dice.

B claims that he will obtain at least two sixes by throwing 12 dice.

C claims that he will obtain at least three sixes by throwing 18 dice.

Which of the three has the greatest probability of success?

Pepys initially thought that outcome C had the highest probability, but Newton correctly showed that outcome A actually has the highest probability.

Although Newton computed the exact probabilities correctly, he also offered Pepys an intuitive explanation. He imagined that B and C were throwing their dice in groups of six, arguing that Aneeded a six in only one group, whereas B and C needed a six in each group. This argument, however, neglects the possibility of obtaining more than one six per group and therefore does not strictly correspond to the original problem.

A detailed analysis and generalization of this problem can be found in [1], but the limiting behavior as the number of dice tends to infinity is not rigorously proved there.The article ends with this conclusion: "As above we can show that $g(\infty, n)$ increases with n and that $f(\infty, n) > g(\infty, n)$, from which we can infer that $g(\infty, n) \to \frac{1}{2}$ as $n \to \infty$ (13)." But rigorous proof of this claim is lacking.

The Newton-Pepys problem has also been examined in [2–4].

Solution and numerical calculations

Let $6.k$ dice be thrown, from the properties of the Bernoulli Scheme the probability of getting at least k sixes for $k = 1, 2, 3, \ldots$ is respectively

$$\begin{aligned} &1 - P_{6k}(0) - P_{6k}(1) - \cdots - P_{6k}(k-1) \\ &= 1 - \left(\frac{5}{6}\right)^{6k} - C_{6k}^{1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{6k-1} - C_{6k}^{2}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{6k-2} \\ &\quad - C_{6k}^{3}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{6k-3} - \cdots - C_{6k}^{k-1}\left(\frac{1}{6}\right)^{k-1}\left(\frac{5}{6}\right)^{5k+1}. \end{aligned}$$

Numerical values computed using Wolfram Mathematica for 6, 12, 18, 24, 36 dice are:

$k = 1$ $P = 0.665102023319615912 21$

$k = 2$ $P = 0.618667373732308713 48$

$k = 3$ $P = 0.597345685947723194 97$

$k = 4$ $P = 0.58448666630151517443$

$k = 5$ $P = 0.57566111636156985379$

$k = 6$ $P = 0.56912439035742736966$

Figure 1 shows the probabilities P_k for $k = 1, 2, \ldots, 170$.

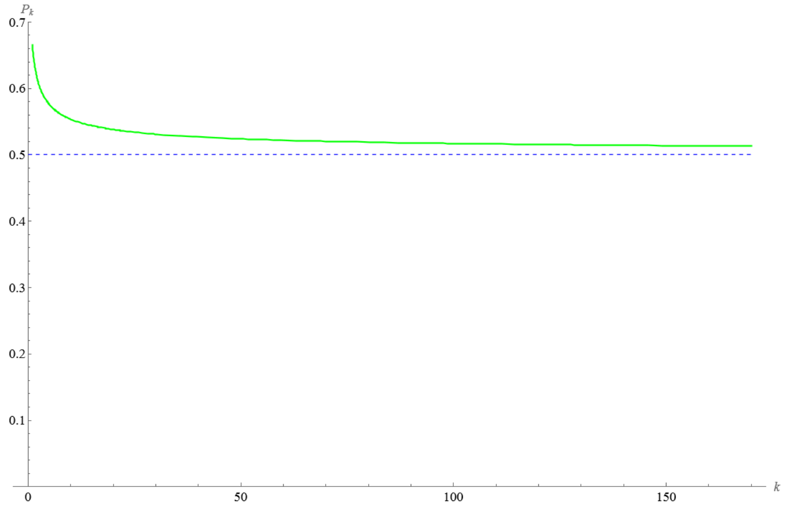

Figure 1. Probability values Pk for k=1,2,3,4,...,170

Generalization to a n-sided die

We now generalize the problem to a die with sides. The generalization of this problem for a "dice" with n sides and $k = 1, 2, 3, \ldots$ is: Let $n.k$ "dice" be thrown, the probability of getting at least k "n"-tuples for $k = 1, 2, 3, \ldots$ is respectively

$$\begin{aligned} &1 - P_{nk}(0) - P_{nk}(1) - \cdots - P_{nk}(k-1) \\ &= 1 - \left(\frac{n-1}{n}\right)^{n.k} - C_{nk}^{1}\left(\frac{1}{n}\right)^{1}\left(\frac{n-1}{n}\right)^{n.k-1} \\ &\quad - C_{nk}^{2}\left(\frac{1}{n}\right)^{2}\left(\frac{n-1}{n}\right)^{n.k-2} - C_{nk}^{3}\left(\frac{1}{n}\right)^{3}\left(\frac{n-1}{n}\right)^{n.k-3} \\ &\quad \cdots - C_{nk}^{k-1}\left(\frac{1}{n}\right)^{k-1}\left(\frac{n-1}{n}\right)^{(n-1)k+1}. \end{aligned}$$

Numerical calculations for $n = 2, 3, \ldots,$ $k = 1, 2, 3, \ldots, n$ made with Wolfram Mathematica and results are listed below:

$n = 2$ $k = 1$ $P = 0.75000000000000000000$

$n = 2$ $k = 2$ $P = 0.68750000000000000000$

$n = 3$ $k = 1$ $P = 0.703703703703703703$

$n = 3$ $k = 2$ $P = 0.64883401920438957476$

$n = 3$ $k = 3$ $P = 0.62282172433064065437$

$n = 4$ $k = 1$ $P = 0.683593750000000000$

$n = 4$ $k = 2$ $P = 0.63291931152343750000$

$n = 4$ $k = 3$ $P = 0.60932499170303344727$

$n = 4$ $k = 4$ $P = 0.59501288994215428829$

$n = 5$ $k = 1$ $P = 0.67232000000000000000$

$n = 5$ $k = 2$ $P = 0.62419036160000000000$

$n = 5$ $k = 3$ $P = 0.60197679074508800000$

$n = 5$ $k = 4$ $P = 0.58855113804343148544$

$n = 5$ $k = 5$ $P = 0.57932569074786841731$

$n = 6$ $k = 1$ $P = 0.66510202331961591221$

$n = 6$ $k = 2$ $P = 0.61866737373230871348$

$n = 6$ $k = 3$ $P = 0.59734568594772319497$

$n = 6$ $k = 4$ $P = 0.58448666630151517443$

$n = 6$ $k = 5$ $P = 0.57566111636156985379$

$n = 6$ $k = 6$ $P = 0.56912439035742736966$

$n = 7$ $k = 1$ $P = 0.66008332291088625609$

$n = 7$ $k = 2$ $P = 0.61485550878898391861$

$n = 7$ $k = 3$ $P = 0.59415718947995952481$

$n = 7$ $k = 4$ $P = 0.58169169290925730915$

$n = 7$ $k = 5$ $P = 0.57314296562891504962$

$n = 7$ $k = 6$ $P = 0.56681446691001593734$

$n = 7$ $k = 7$ $P = 0.56188675551612881974$

$n = 8$ $k = 1$ $P = 0.65639108419418334961$

$n = 8$ $k = 2$ $P = 0.61206528550161110047$

$n = 8$ $k = 3$ $P = 0.59182715192826333793$

$n = 8$ $k = 4$ $P = 0.57965091988117587949$

$n = 8$ $k = 5$ $P = 0.57130522754214376694$

$n = 8$ $k = 6$ $P = 0.56512924758534476526$

$n = 8$ $k = 7$ $P = 0.56032149826338641063$

$n = 8$ $k = 8$ $P = 0.55644169917396053505$

$n = 9$ $k = 1$ $P = 0.65356058388538144662$

$n = 9$ $k = 2$ $P = 0.60993412562702706743$

$n = 9$ $k = 3$ $P = 0.59004963334090925546$

$n = 9$ $k = 4$ $P = 0.57809500929920911203$

$n = 9$ $k = 5$ $P = 0.56990461716802312983$

$n = 9$ $k = 6$ $P = 0.56384518477273793430$

$n = 9$ $k = 7$ $P = 0.55912904466766924123$

$n = 9$ $k = 8$ $P = 0.55532369839480008392$

$n = 9$ $k = 9$ $P = 0.55216938304783993682$

$n = 10$ $k = 1$ $P = 0.65132155990000000000$

$n = 10$ $k = 2$ $P = 0.60825300187483229419$

$n = 10$ $k = 3$ $P = 0.58864876044049461775$

$n = 10$ $k = 4$ $P = 0.57686934690699793228$

$n = 10$ $k = 5$ $P = 0.56880159317093831426$

$n = 10$ $k = 6$ $P = 0.56283412926287325816$

$n = 10$ $k = 7$ $P = 0.55819024277035292231$

$n = 10$ $k = 8$ $P = 0.55444359756159237871$

$n = 10$ $k = 9$ $P = 0.55133819632449137099$

$n = 10$ $k = 10$ $P = 0.54870983455799641343$

We observe that for fixed n, the probability is maximal for $k = 1$ and decreases monotonically as k increases, with the maximum value being about $\frac{2}{3}$ and actually tends to

$$1 - e^{-1} = 0.63212055882855767840,$$

while the minimal value (attained at) is always greater than $\frac{1}{2}$.

A large number of similar tests were conducted by the author. In all tests conducted, the hypothesis that the probability decreases and tends to $\frac{1}{2}$ as $k \to \infty$ is confirmed.

For example, for $n = 1, 2, 3, 4, \ldots, 99, 100$ the probability for $k = 1$ monotonically decreases and for $n = 100$ for example is

$$p = 0.63396765872677049507,$$

and the smallest probability is for $k = n = 100$:

$$P = 0.51349927171675474161 > 0.5$$

when we stop at $k = n$.

Unfortunately it is not possible using Wolfram Mathematica to prove that the limit is $\frac{1}{2}$. The author has not found theoretical proof for the claim that the limit is $\frac{1}{2}$. Below we will prove this claim using the Central Limit Theorem.

Limiting behavior and marginal probabilities

First, consider the case $k = 1$. We will show that as n increases, the probability of having at least 1 success in n trials ($k = 1$) tends to $1 - e^{-1}$. Indeed, this probability is

$$1 - P(0) = 1 - \left(\frac{n-1}{n}\right)^n = 1 - \left(1 - \frac{1}{n}\right)^n \to 1 - e^{-1}.$$

Now consider the case of large k. In this case the number of successes is

$$\eta = \xi_1 + \xi_2 + \ldots + \xi_{n.k}, \quad \xi_i \sim Be\left(\frac{1}{n}\right)$$

and are independent. Then

$$E(\eta) = n.k, \quad E(\xi_i) = n.k.\tfrac{1}{n} = k,$$
$$D(\eta) = n.k.D(\xi_i) = n.k.\frac{1}{n}.\left(1 - \frac{1}{n}\right) = k.\left(1 - \frac{1}{n}\right).$$

The conditions of the central limit theorem are satisfied for large k and η will be approximately normally distributed: $\eta \sim N\left(k, k.(1 - \frac{1}{n})\right)$. Then $P(\eta \geq k) \to 0.5$ by the properties of the normal distribution.

Poisson approximation

For large n and fixed k, the distribution of η converges to a Poisson distribution with parameter k:

Let's consider the cases of large n and fixed k. For small values of j we get

$$P(\eta = j) = C_{nk}^{j}\left(\frac{1}{n}\right)^{j}\left(\frac{n-1}{n}\right)^{nk-j}$$

$$= \frac{(nk)!}{j!(nk-j)!}\cdot\frac{1}{n^j}\cdot\left(1-\frac{1}{n}\right)^{nk}\left(\frac{n}{n-1}\right)^{j}$$

$$= \left(\left(1-\frac{1}{n}\right)^{n}\right)^{k}\cdot\frac{k^j}{j!}\cdot\frac{(nk)(nk-1)\cdots(nk-j+1)}{(nk)^j}\cdot\left(\frac{n}{n-1}\right)^{j}$$

$$\to\infty\ e^{-k}\frac{k^j}{j!}\ \text{ when } n\to\infty.$$

Therefore $\eta \to Po(k)$ when $n \to \infty$ and if $k = n$ then

$$S(n) = e^{-n}.\left(1+\frac{n}{1!}+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}\right) = P(\eta\le n)\to\frac{1}{2}.$$

This is another limit problem for which the author does not know of a solution that does not use the central limit theorem.

Conclusions

We have analyzed a generalization of the Newton–Pepys problem to n-sided dice and provided a rigorous proof using the Central Limit Theorem that, for fixed n, the probability of obtaining at least k successes in nk trials converges to $\frac{1}{2}$ as $k \to \infty$.

This result explains the monotonic decrease observed in numerical computations and clarifies the asymptotic behavior previously left unproven in the literature. The analysis also highlights a general probabilistic principle: strategies requiring fewer successes over fewer trials tend to have higher probabilities of success than strategies requiring many successes over many trials with the same expected number of successes.

A good illustration of the usefulness of these considerations is the following problem, which in a sense is a summary of the Pepys-Newton problem:

In a game we can make "big" or "small" bets. With the money we have, we can make, for example, 10 "big" or 100 "small" bets, and in order to make a profit we must succeed at least once with a "big" bet or at least 10 times with a "small". The probability of success with each bet is the same and is equal to 0.1. Which strategy is better – 10 "big" or 100 "small" bets?

From the calculations above, the probability of at least one "success" in 10 bets will be $P = 0.6513215599$, and for at least 10 "successes" in 100 bets will be $P = 0.5426994078$.

The conclusion from the reasoning and calculations made above is that it is more profitable to play with 10 "big" bets rather than with 100 "small". That is, if you are, for example, a venture investor who has to choose between investing in 10 large projects or alternatively in 100 small projects under the above conditions, the more profitable choice in terms of odds is the one in 10 large projects.

This conclusion is obviously summarized in the rule: a better strategy is to play with a smaller number of larger bets than with a larger number of smaller ones under similar winning conditions to the above. For "short" bets, the probability under these conditions is close to $1-e^{-1} = 0.63212055882855767840$, and for "long" bets it is close to 0.5. For other probabilities and lengths of the series, these probabilities are different, but the monotonous decrease in probabilities that we have noticed shows that the "short" strategy is the more profitable.

It is interesting to make statistics on how real investors choose? Are there statistics on their choices and are there real situations in which a result similar to our calculations was obtained?

References

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